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ECON61001 Econometric Methods

Lecture 2

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Matrices - revision

- matrices were mentioned in PreSession Maths:
- A matrix is a rectangular array of numbers enclosed in parentheses,
- conventionally denoted by a capital letter.
- the number of rows (say m) and the number of columns (say n) determine the order of the matrix ($m \times n$).
- examples:
- $P = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 1 & 5 \end{bmatrix}$
- 2×3 and 3×2 respectively

Matrices and Econometrics

- data sets are matrices ...
- here observations on
- weights and heights of 12 students

$$D = \begin{bmatrix} 155 & 70 \\ 150 & 63 \\ 180 & 72 \\ 135 & 60 \\ 156 & 66 \\ 168 & 70 \\ 178 & 74 \\ 160 & 65 \\ 132 & 62 \\ 145 & 67 \\ 139 & 65 \\ 152 & 68 \end{bmatrix}$$

Matrix Arithmetic and Matrix Algebra

- calculations using matrices with numerical elements is *matrix arithmetic*
- calculations using matrices with symbolic elements is *matrix algebra*
- e.g with $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ or $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- general 2×3 and $m \times n$ matrices
- really want to use the algebra of matrices
- that is algebra with objects that are matrices
- rather than algebra with the elements of matrices
- start with matrix arithmetic
- and build up to the two versions of matrix algebra

Typical element notation for matrices

- for $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad m \times n$
- write $A = \|a_{ij}\|, \qquad i = 1, ..., m, j = 1, ..., n$
- a_{ij} is the element in (at the intersection of) the ith row and jth column, e.g. a_{12}
- when $m \neq n$, A is a *rectangular* matrix
- when $m = n$, A is $m \times m$ or $n \times n$, and A is a square matrix
- so a square matrix has the same number of rows and columns

Rows, columns and vectors

- if A is $m \times n$, $m = 1$ or $n = 1$ or both is allowed
- if $n = 1$, say that A is an $m \times 1$ column vector
- $A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$
- if $m = 1$, A is a $1 \times n$ row vector
- $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \end{bmatrix}$
- usual to use bold lower case for vectors
- e.g. $\mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$
- if $m = 1 = n$, $A = [a_{11}] = a_{11}$ - both a 1×1 matrix and a real number

Matrices as collections of vectors

- think of $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ as a collection of columns
- each column is a column vector (or just a *vector)*
- e.g. $\mathbf{a} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, 2×1 vectors
- define $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$, a 2×2 matrix

Transposition of vectors

- rows of $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$ are vectors
- $\mathbf{c} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ... but these are column vectors, not rows
- convert a column vector **c** into a row vector by transposition
- the transposed **c** is $\mathbf{c}^T = \begin{bmatrix} 6 & 2 \end{bmatrix}$
- here ^{T} denotes transposition
- sometimes write $\mathbf{c}'$ - i.e. use a prime, but easier to lose track of $'$ in calculations
- stick to the ^{T} sign!
- write A in terms of its rows as $A = \begin{bmatrix} \mathbf{c}^T \\ \mathbf{d}^T \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$
- note that the transpose of $\mathbf{c}^T$ is $\mathbf{c}$: $\left(\mathbf{c}^T\right)^T = \mathbf{c}$

Operations with vectors

- set $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $n \times 1$ (column) vectors
- addition and subtraction defined only for vectors of the same dimensions
- $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad \mathbf{x} - \mathbf{y} = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix}$
- these operations are *elementwise*
- if $\mathbf{x}$ and $\mathbf{y}$ had different dimensions, there would be some elements left over from the larger dimension vector

Scalar multiplication

- for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- if λ is a real number or *scalar*, the product $\lambda\mathbf{x}$ is defined as
- $\lambda\mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$
- every element of $\mathbf{x}$ is multiplied by λ to give $\lambda\mathbf{x}$

Linear combinations of vectors

- addition of vectors and scalar multiplication can be combined to give
- a linear combination of $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$,
- as $\lambda \mathbf{x} + \mu \mathbf{y} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} + \begin{bmatrix} \mu y_1 \\ \vdots \\ \mu y_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 + \mu y_1 \\ \vdots \\ \lambda x_n + \mu y_n \end{bmatrix}$
- more generally
- the linear combination of vectors $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ by scalars $\lambda, \mu, \ldots, \nu$ is
- $\lambda \mathbf{x} + \mu \mathbf{y} + \ldots + \nu \mathbf{z}$
- with typical element $\lambda x_i + \mu y_i + \ldots + \nu z_i$
- $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ must have a common dimension

Linear combinations of matrices

- carry over to matrices - apply to each column of a matrix
- for $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$, both $m \times n$
- $A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \dots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix} = \|a_{ij} + b_{ij}\|$
- $A - B = \begin{bmatrix} \mathbf{a}_1 - \mathbf{b}_1 & \dots & \mathbf{a}_n - \mathbf{b}_n \end{bmatrix} = \|a_{ij} - b_{ij}\|$
- so addition/subtraction is really elementwise
- scalar multiplication of A by λ is also elementwise
- $\lambda A = \begin{bmatrix} \lambda\mathbf{a}_1 & \dots & \lambda\mathbf{a}_n \end{bmatrix} = \|\lambda a_{ij}\|$
- the linear combination of A and B by λ and μ is
- $\lambda A + \mu B = \begin{bmatrix} \lambda\mathbf{a}_1 + \mu\mathbf{b}_1 & \dots & \lambda\mathbf{a}_n + \mu\mathbf{b}_n \end{bmatrix} = \|\lambda a_{ij} + \mu b_{ij}\|$

Example

- $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $\lambda = 1,\ \mu = -2$
- then
- $\lambda A + \mu B = A - 2B = \begin{bmatrix} 4 & 0 \\ 1 & 7 \end{bmatrix}$

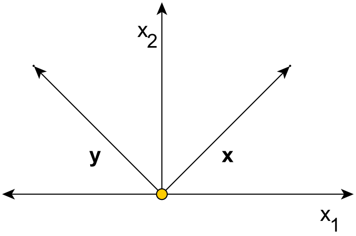
Inner products

- for two vectors **a** and **x**, with **a** written as a row vector,
- $\mathbf{a}^T = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- the product $\mathbf{a}^T\mathbf{x}$ is called the *inner product*
- it is defined as $\mathbf{a}^T\mathbf{x} = a_1x_1 + \ldots + a_nx_n$
- usually called the *across and down rule*
- multiply together corresponding elements in $\mathbf{a}^T$ and **x**, and add up the products
- result of $\mathbf{a}^T\mathbf{x}$ is a real number
- e.g.
  $\mathbf{c}^T = \begin{bmatrix} 6 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $\mathbf{c}^T\mathbf{x} = \begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 36 + 6 = 42$

- for $\mathbf{a}^T\mathbf{x}$ to be defined, $\mathbf{a}$ and $\mathbf{x}$ must both be $n \times 1$
- so for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$
- $\mathbf{b}^T\mathbf{x}$ is not defined

Orthogonality

- if $\mathbf{x}$ and $\mathbf{y}$ are such that
- $\mathbf{x}^T\mathbf{y} = 0$,
- $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other
- e.g. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x}^T\mathbf{y} = 0$
- arrows represent the vectors
- the vectors are at right angles to each other



Matrix vector products

- write $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \alpha_1^T \\ \alpha_2^T \end{bmatrix}$ i.e. through its rows
- given $\mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, two possible inner products,
- $\alpha_1^T \mathbf{x} = 42, \qquad \alpha_2^T \mathbf{x} = 33$
- assemble into 2×1 vector - defines the product $A\mathbf{x}$
- $A\mathbf{x} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \alpha_1^T \mathbf{x} \\ \alpha_2^T \mathbf{x} \end{bmatrix} = \begin{bmatrix} 42 \\ 33 \end{bmatrix}$
- numerically, $\begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 42 \\ 33 \end{bmatrix}$
- is an *across and down* rule
- notice that the row dimension of $A\mathbf{x}$ is that of A

Linear combinations of columns

- another perspective on $A\mathbf{x} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 42 \\ 33 \end{bmatrix}$
- $A\mathbf{x} = 6 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is a linear combination of the columns of A
- more generally, if $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$
- then $A\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b}$
- even more generally ... an $m \times n$ matrix A is a collection of columns,
- $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$

- If $\mathbf{x}$ is an $n \times 1$ vector, by the *across and down* rule,
- $$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
- $$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ a_{21}x_1 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix}$$
- the ith element of $A\mathbf{x}$ is $\sum_{j=1}^{n} a_{ij}x_j$
- $A\mathbf{x}$ is an $m \times 1$ vector

- as a linear combination of the columns of A, where
- $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- $A\mathbf{x} = \mathbf{a}_1 x_1 + \ldots + \mathbf{a}_n x_n = \sum_{j=1}^{n} \mathbf{a}_j x_j = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$

Matrix - matrix products

- $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix}$, $m \times n$, $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_r \end{bmatrix}$, $n \times r$
- each $A\mathbf{b}_i$ exists and is $m \times 1$
- arrange products as columns of matrix $C = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_r \end{bmatrix}$
- define this matrix C as the product AB, an $m \times r$ matrix
- and create an across and down rule for defining C
- $B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nr} \end{bmatrix} = \left\| b_{ik} \right\|, \qquad i = 1, ..., n, k = 1, ..., r$
- the kth column of C is $A\mathbf{b}_k$
- typical element of C is obtained as inner product of ith row of A
- $\begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix}$, with the elements of $\mathbf{b}_k$
- $a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$

- $a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$
- is the ikth element of C
- arises from the across and down argument in
- $C = AB =$
$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1k} & \ldots & b_{1r} \\ b_{21} & b_{22} & \ldots & b_{2k} & \ldots & b_{2r} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nk} & \ldots & b_{nr} \end{bmatrix}$$
- in typical element notation, $C = \left\| \sum_{j=1}^{n} a_{ij}b_{jk} \right\|$
- simple ideas, but a lot of detail, numerical examples inevitably tedious
- need to do hand calculations to start with
- but end up by using computer - Matlab

Examples

- $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}$
- sum and difference of $\mathbf{x}$ and $\mathbf{y}$ as matrix vector products are
- $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$
- $\mathbf{x} - \mathbf{y} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$
- set $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ to contain the linear combination coefficients
- then $C = AB = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 8 & -2 \end{bmatrix}$

Conformable Matrices

- if A is $m \times n$, B must be $n \times r$ for the product AB to be defined,
- so that $m \times n : n \times r$ produces an $m \times r$ matrix
- say that A and B are conformable in this case
- key point is that if the number of columns of A is not equal to the number of rows of B, then the inner product calculations required are not defined

Overview

- matrix arithmetic viewpoint good to see the ideas working
- but the elementwise approach to matrix multiplication is not good for
- matrix algebra
- the linear combination of columns perspective is much more useful
- note the conformability requirement
- for AB to be defined,
- A must have the same number of columns
- as there are rows in B
- Matlab is very useful for these matrix calculations - lecture notes give some examples

Pre and post multiplication

- If $C = AB$, B is pre-multiplied by A, and A is post-multiplied by B
- suppose that AB and BA are both defined
- if A is $m \times n$, B must be $n \times r$ to get AB, $m \times r$
- to get BA with A $m \times n$, B must be $n \times m$ $-$ i.e. $r = m$
- AB is then $m \times m$, BA is $n \times n$
- but different sized matrices cannot be equal
- e.g. $B_2 C = \begin{bmatrix} 6 & -3 \\ 2 & 5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & -3 \\ 3 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 27 & -3 & -15 \\ 27 & 29 & -11 \\ -15 & -1 & 8 \end{bmatrix}$
- $CB_2 = \begin{bmatrix} 6 & 2 & -3 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 49 & -11 \\ 31 & 15 \end{bmatrix}$

- even when $m = n$ so that AB and BA are both $m \times m$
- AB and BA are not necessarily equal
- e.g. $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $AB = \begin{bmatrix} 8 & 4 \\ 8 & -2 \end{bmatrix}, \qquad BA = \begin{bmatrix} 9 & 7 \\ 3 & -3 \end{bmatrix}$
- in cases where $AB = BA$, A and B are said to commute

Transposition

- convert column vector $\mathbf{x}$ to row vector $\mathbf{x}^T$ by transposition
- $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{x}^T = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$
- transpose $\mathbf{x}^T$ as $\left(\mathbf{x}^T\right)^T$ to recover $\mathbf{x}$
- for an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$ the transpose of A, A^T, is the matrix whose rows are the columns of A transposed
- $A^T = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}, \, n \times m$

- if the rows of A^T are transposed columns of A ...
- then, elementwise, $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$,
- is transposed to $A^T = \begin{bmatrix} a_{11} & \dots & a_{i1} & \dots & a_{m1} \\ a_{12} & \dots & a_{i2} & \dots & a_{m2} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \dots & a_{in} & \dots & a_{mn} \end{bmatrix}$
- so (i, j) element of A is the (j, i) element of A^T
- what about transposing A^T?
- write rows as columns so that $\left(A^T\right)^T = A$

Product rule for transposition

- ... states that if $C = AB$, then $C^T = B^T A^T$, example 'proof' in lecture notes
- to transpose AB, transpose terms from right to left
- e.g. $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$, $A^T = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}$
- $B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$
- $C = AB = \begin{bmatrix} 10 & 4 \\ 13 & -2 \end{bmatrix}$, $C^T = \begin{bmatrix} 10 & 13 \\ 4 & -2 \end{bmatrix}$
- $B^T A^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 13 \\ 4 & -2 \end{bmatrix}$

Coordinate vectors

- vectors of the form
- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in 2 dimensions, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in 3 dimensions
- $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ in n dimensions
- are called coordinate vectors
- characteristic notation, $\mathbf{e}_1, \ldots, \mathbf{e}_n$, in n dimensions
- also a characteristic pattern of elements

Zero and identity matrices

- the zero matrix has every element equal to zero: $0 = \|0\|$
- but what is the dimension? if $m \times n$, can write 0_{mn} - but usually omitted
- effects: turns any matrix into the zero matrix, $0A = 0, \qquad B0 = 0$
- identity or unit matrix is formed from coordinate vectors
- 2 dimensions: $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$
- 3 dimensions: $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

- n dimensions: $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_n$
- characteristic pattern of $1'$s on the diagonal, zeros elsewhere
- effects? use $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$
- $I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = A$
- $A I_2 = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = A$
- any matrix is left unchanged by pre or post multiplication by a suitable I_n
- hence the name identity matrix, always a square matrix

Diagonal matrices

- diagonal matrix: every element zero except on the diagonal
- usually square, e.g. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- characteristic effects ... $A = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$
- $AB = \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ 6 & -10 \end{bmatrix}$
- $BA = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ -6 & -10 \end{bmatrix}$
- post multiplication multiplies each column of A by the corresponding diagonal element
- pre multiplication multiplies each row by the corresponding diagonal element

Symmetric matrices

- A is symmetric if $A = A^T$, so a symmetric matrix must be square
- e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$
- equality of matrices is equality of all elements - ok on the diagonal
- for the off diagonal elements, must have
- $a_{12} = a_{21}, \quad a_{13} = a_{31}, \quad a_{23} = a_{32}$
- more generally, $a_{ij} = a_{ji} \qquad \text{for} \quad i \neq j$
- for a symmetric matrix, the triangle of above diagonal elements coincides with the triangle of below diagonal elements
- e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A common symmetric matrix

- e.g $C = \begin{bmatrix} 6 & 2 & -3 \\ 3 & 5 & -1 \end{bmatrix}$, compute $C^T C$
- $C^T C = \begin{bmatrix} 6 & 3 \\ 2 & 5 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & -3 \\ 3 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 45 & 27 & -21 \\ 27 & 29 & -11 \\ -21 & -11 & 10 \end{bmatrix}$
- general result here: if A is $m \times n$, then $A^T A$ is symmetric, $n \times n$
- proof using product rule for transposition
- $\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$
- such symmetric matrices appear a lot in econometrics
- can see that all diagonal matrices are symmetric

The outer product

- $\mathbf{x}$, $\mathbf{y}$, $n \times 1$, inner product $\mathbf{x}^T\mathbf{y}$ is 1×1, a scalar
- what about the product $\mathbf{x}\mathbf{x}^T$? $n \times 1 : 1 \times n$ should give $n \times n$?
- i.e. an $n \times n$ matrix
- what does the across and down rule say?
- e.g. $\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \begin{bmatrix} 6 & 3 \end{bmatrix}$
- $\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 36 & 18 \\ 18 & 9 \end{bmatrix}$, a symmetric matrix

- if $\mathbf{x}$ is $n \times 1$ and $\mathbf{y}$ is $m \times 1$, $\mathbf{x}\mathbf{y}^T$ is $n \times m$
- $\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \dots & y_m \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \dots & x_2 y_m \\ & & \vdots & \\ x_n y_1 & x_n y_2 & \dots & x_n y_m \end{bmatrix}$
- examples with $\mathbf{1}$, a vector with every element 1, are interesting
- $\mathbf{1}$ is often called the sum vector
- $\mathbf{1}_2^T \mathbf{x} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 9$, the sum of the elements of $\mathbf{x}$
- easy to turn into sample mean of elements of $\mathbf{x}$
- ... divide by number of elements in $\mathbf{x}$

- outer products with $\mathbf{1}$...
- $\mathbf{1}_2\mathbf{x}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 6 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 6 & 3 \end{bmatrix}$
- $\mathbf{x}\mathbf{1}_2^T = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 3 \end{bmatrix}$
- so premultiplication by $\mathbf{1}$ repeats $\mathbf{x}^T$ as rows of product
- postmultiplication by $\mathbf{1}$ repeats $\mathbf{x}$ as columns of product
- notice that $\mathbf{1}_n\mathbf{1}_n^T = \begin{bmatrix} 1 & \ldots & 1 \\ & \vdots & \\ 1 & \ldots & 1 \end{bmatrix}$ - useful in econometrics

Triangular matrices

- $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a lower triangular matrix because all elements above the diagonal are zero
- lower triangular matrices are usually square, but rectangular versions permitted
- A^T is an upper triangular matrix, with all elements below the diagonal zero
- unit triangular matrices have diagonal elements all equal to 1
- e.g. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Partitioned matrices

- can be helpful organise blocks of elements of a matrix into matrices
- e.g.

$$B = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ \hline 0 & 0 & 7 & 4 \\ 0 & 0 & 6 & 5 \end{array} \right] = \left[\begin{array}{cc} B_{11} & 0 \\ 0 & B_{22} \end{array} \right]$$

- where the blocks are 2×2 matrices
- B is a *partitioned matrix*

More examples

- A, $m \times n$ with $A = \begin{bmatrix} A_{11} & A_{12} & A_{23} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$
- a partition of A into r rows and $m - r$ rows,
- 3 columns, 4 columns and $n - 7$ columns
- $A_{11} : r \times 3$, $A_{12} : r \times 4$, $A_{13} : r \times (n - 7)$
- $A_{21} : (m - r) \times 3$, $A_{22} : (m - r) \times 4$, $A_{23} : (m - r) \times (n - 7)$
- another example: $A = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}$, $\quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$
- A is $m \times (n_1 : n_2 : n_3)$
- for $A\mathbf{x}$ to exist, $\mathbf{x}_1$ must be $n_1 \times 1$, $\mathbf{x}_2$ $n_2 \times 1$ and $\mathbf{x}_3$ $n_3 \times 1$
- then $A_1\mathbf{x}_1$, $A_2\mathbf{x}_2$ and $A_3\mathbf{x}_3$ are all $m \times 1$

- if A is $m \times n$ and $\mathbf{x}$ $n \times 1$, typical element of $A\mathbf{x}$ is $\sum_{j=1}^{n} a_{ij}x_j$
- by the across and down rule, break up summation into three components
- the part from the first n_1 columns of A, $\sum_{j=1}^{n_1} a_{ij}x_j$, corresponds to $A_1\mathbf{x}_1$
- the part from the next n_2 columns of A, $\sum_{j=n_1+1}^{n_1+n_2} a_{ij}x_j$, corresponds to $A_2\mathbf{x}_2$
- the part from the last n_3 columns of A, $\sum_{j=n_1+n_2+1}^{n} a_{ij}x_j$, corresponds to $A_3\mathbf{x}_3$
- clear that $A\mathbf{x} = A_1\mathbf{x}_1 + A_2\mathbf{x}_2 + A_3\mathbf{x}_3$
- a natural generalisation of the across and down rule
- but each component product has to be conformable

- another example with $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix}$
- A submatrix dimensions
- $\begin{bmatrix} r \times 3 & r \times 4 & r \times (n-7) \\ (m-r) \times 3 & (m-r) \times 4 & (m-r) \times (n-7) \end{bmatrix}$
- $AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix}$
- $AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \end{bmatrix}$
- what must the dimensions of the submatrices in B be for this to be defined?
- B_{11}, B_{21}, B_{31} must have the same number of columns
- B_{11} must have 3 rows, B_{21} 4 rows and B_{31} $n-7$ rows

Matrices, vectors and econometrics

- regress weight on height: $y_i = \alpha + \beta x_i + u_i$,
- think of D as $D = \begin{bmatrix} \mathbf{y} & \mathbf{x} \end{bmatrix}$ say,
- define $\mathbf{1}_{12} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{12} \end{bmatrix}$
- $\mathbf{u}$ a 12×1 vector of error terms
- regression model is $\mathbf{y} = \mathbf{1}_{12}\alpha + \mathbf{x}\beta + \mathbf{u}$
- combine components:

  $X = \begin{bmatrix} \mathbf{1}_{12} & \mathbf{x} \end{bmatrix}$, $\boldsymbol{\delta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$
- data matrix representation is $\mathbf{y} = X\boldsymbol{\delta} + \mathbf{u}$

$$D = \begin{bmatrix} 155 & 70 \\ 150 & 63 \\ 180 & 72 \\ 135 & 60 \\ 156 & 66 \\ 168 & 70 \\ 178 & 74 \\ 160 & 65 \\ 132 & 62 \\ 145 & 67 \\ 139 & 65 \\ 152 & 68 \end{bmatrix}$$